

# Einstein's Field Equations for the Interior of a Uniformly Rotating Stationary Axisymmetric Perfect Fluid

E. Kyriakopoulos  
Department of Physics  
National Technical University  
157 80 Zografou, Athens, GREECE

## Abstract

We reduce Einstein's field equations for the interior of a uniformly rotating, axisymmetric perfect fluid to a system of six second order partial differential equations for the pressure  $p$  the energy density  $\mu$  and four dependent variables. Four of these equations do not depend on  $p$  and  $\mu$  and the other two determine  $p$  and  $\mu$ .

PACS number(s): 04.20.Jb, 04.20.-q

## 1 Introduction

The number of solutions of Einstein's field equations for the interior of a uniformly rotating stationary axisymmetric perfect fluid is very limited contrary to what happens in the stationary axisymmetric vacuum case [1]. This is mainly due to the fact that the equations we have to solve in the first case are complicated [2], while in the second case they have been reduced to the equivalent and much simpler Ernst's equation [3]. Reduction of the "interior" problem is known for Einstein-Maxwell field [4] and dust [5]. Also Einstein's field equations for the interior of a uniformly rotating stationary axisymmetric perfect fluid have been reduced to a system of two second order partial differential equations for two unknown functions, which however are very complicated [6].

In this work the six Einstein's field equations for the interior of a uniformly rotating stationary axisymmetric perfect fluid, which depend on the pressure  $p$  the energy density  $\mu$  of the fluid and on four dependent variables, will be divided into a set of four equations which depend on the four dependent variables but not on  $p$  and  $\mu$  and a set of two equations which give  $p$  and  $\mu$  as functions of the four dependent variables. Therefore solving the

first system of equations we can determine  $p$  and  $\mu$  algebraically using the second set. Introducing a potential and redefining variables we write the first system in a relatively simple form, which is very convenient for finding the Bäcklund transformations the equations of the system may posses. Also we eliminate one of the dependent variables of the system increasing however the complexity of the problem.

## 2 The Field Equations

The line element of stationary axisymmetric fields admitting 2-spaces orthogonal to the Killing vectors  $\vec{\xi} = \partial_t$  and  $\vec{\eta} = \partial_\phi$  can be written in the form [7], [1]

$$ds^2 = e^{-2U} \{e^{2K}(d\rho^2 + dz^2) + F^2 d\phi^2\} - e^{2U}(dt + Ad\phi)^2 \quad (1)$$

where  $U = U(\rho, z)$ ,  $K = K(\rho, z)$ ,  $F = F(\rho, z)$  and  $A = A(\rho, z)$ . The decomposition of the metric into orthogonal 2-spaces whose points are labeled by  $\rho$  and  $z$  is possible for perfect fluid solutions provided that the 4-velocity of the fluid satisfies the circularity condition  $u_{[a}\xi_b\eta_{c]} = 0$  [1].

For a perfect fluid source we have the energy-momentum tensor

$$T_{ab} = (\mu + p)u_a u_b + pg_{ab} \quad (2)$$

where  $u_a$  are the components of the 4-velocity and  $\mu$  and  $p$  are the mass density and the pressure of the fluid respectively. We shall introduce the notation

$$\partial_\rho = \frac{\partial}{\partial \rho}, \partial_z = \frac{\partial}{\partial z}, \partial = \frac{1}{2}(\partial_\rho - i\partial_z), \bar{\partial} = \frac{1}{2}(\partial_\rho + i\partial_z). \quad (3)$$

Then Einstein's field equations for a uniformly rotating axisymmetric perfect fluid can be written in the form [2]

$$2\partial\bar{\partial}F = pFe^{2K-2U} \quad (4)$$

$$2\partial\bar{\partial}U + \frac{1}{F}(\partial U\bar{\partial}F + \bar{\partial}U\partial F) + \frac{1}{F^2}e^{4U}\partial A\bar{\partial}A = \frac{1}{4}(\mu + 3p)e^{2K-2U} \quad (5)$$

$$2\partial\bar{\partial}A - \frac{1}{F}(\partial A\bar{\partial}F + \bar{\partial}A\partial F) + 4(\partial A\bar{\partial}U + \bar{\partial}A\partial U) = 0 \quad (6)$$

$$2\bar{\partial}F\bar{\partial}K - \bar{\partial}\bar{\partial}F - 2F\bar{\partial}U\bar{\partial}U + \frac{1}{2F}e^{4U}\bar{\partial}A\bar{\partial}A = 0 \quad (7)$$

$$\partial\bar{\partial}K + \partial U\bar{\partial}U + \frac{1}{4F^2}e^{4U}\partial A\bar{\partial}A = \frac{1}{4}pe^{2K-2U} \quad (8)$$

Also from the conservation relation  $T_{;b}^{ab} = 0$  we get equations

$$\partial p + (\mu + p)\partial U = 0 \quad (9)$$

which are obtained from Eq. (4) - (8) and will be omitted in the following.

Eqs (4), (5) and (8) are equivalent to the system

$$p = \frac{2}{F}\partial\bar{\partial}Fe^{2U-2K} \quad (10)$$

$$\begin{aligned} \mu = e^{2U-2K}\{ & 8\partial\bar{\partial}U + \frac{4}{F}(\partial U\bar{\partial}F + \bar{\partial}U\partial F) + \frac{4}{F^2}e^{4U}\partial A\bar{\partial}A \\ & -12\partial\bar{\partial}K - 12\partial U\bar{\partial}U - \frac{3}{F^2}e^{4U}\partial A\bar{\partial}A\} \end{aligned} \quad (11)$$

$$4\partial\bar{\partial}K + 4\partial U\bar{\partial}U + \frac{1}{F^2}e^{4U}\partial A\bar{\partial}A - \frac{2}{F}\partial\bar{\partial}F = 0 \quad (12)$$

We shall replace Eqs (4), (5) and (8) by Eqs (10) - (12).

If we put

$$A = -\omega, \quad e^{2U} = T \quad (13)$$

Eq (6) takes the form

$$\partial(\frac{T^2}{F}\bar{\partial}\omega) + \bar{\partial}(\frac{T^2}{F}\partial\omega) = 0 \text{ or } \partial_\rho(\frac{T^2}{F}\omega_\rho) + \partial_z(\frac{T^2}{F}\omega_z) = 0 \quad (14)$$

where a letter as an index in a function means differentiation with respect to the corresponding variable e.g.  $\omega_\rho = \frac{\partial\omega}{\partial\rho}$ . The solution of the above equation is

$$\frac{T^2}{F}\partial\omega = i\partial\phi \quad (15)$$

or

$$\omega_\rho = \frac{F}{T^2}\phi_z \text{ and } \omega_z = -\frac{F}{T^2}\phi_\rho \quad (16)$$

where  $\phi$  is an arbitrary function. Also we have

$$\begin{aligned} T\{\phi_{\rho\rho} + \phi_{zz} + \frac{1}{F}(F_\rho\phi_\rho + F_z\phi_z)\} - 2(T_\rho\phi_\rho + T_z\phi_z) \\ = \frac{T^3}{F}\{\partial_\rho(\frac{F}{T^2}\phi_\rho) + \partial_z(\frac{F}{T^2}\phi_z)\} \end{aligned} \quad (17)$$

But if Eqs (16) are satisfied the above relation vanish. Therefore we can substitute Eq (14) by the relation

$$T\{\phi_{\rho\rho} + \phi_{zz} + \frac{1}{F}(F_{\rho}\phi_{\rho} + F_z\phi_z)\} - 2(T_{\rho}\phi_{\rho} + T_z\phi_z) = 0 \quad (18)$$

and Eqs (16) which define  $\omega$ . Then using Eqs (16) to eliminate  $\omega$  and introducing the Ernst's potential

$$E = T + i\phi \quad (19)$$

Eqs (7), (10) - (12) and (18) become

$$Im\{\frac{1}{2}(E + \bar{E})(\nabla^2 E + \frac{1}{F}\vec{\nabla}F \cdot \vec{\nabla}E) - \vec{\nabla}E \cdot \vec{\nabla}E\} = Im\Lambda = 0 \quad (20)$$

$$2\bar{\partial}F\bar{\partial}K - \bar{\partial}\bar{\partial}F - \frac{2F}{(E + \bar{E})^2}\bar{\partial}E\bar{\partial}\bar{E} = 0 \quad (21)$$

$$2\partial\bar{\partial}K - \frac{1}{F}\partial\bar{\partial}F + \frac{1}{(E + \bar{E})^2}(\partial E\bar{\partial}\bar{E} + \bar{\partial}E\partial\bar{E}) = 0 \quad (22)$$

$$p = \frac{1}{F}(E + \bar{E})e^{-2K}\partial\bar{\partial}F \quad (23)$$

$$\mu = e^{-2K}(\frac{2}{E + \bar{E}}Re\Lambda - N) = -3p + \frac{2}{E + \bar{E}}e^{-2K}Re\Lambda \quad (24)$$

where

$$\nabla^2 = \partial_{\rho\rho}^2 + \partial_{zz}^2, \quad \vec{\nabla} = \hat{\rho}\partial_{\rho} + \hat{z}\partial_z \quad (25)$$

$$\Lambda = \frac{1}{2}(E + \bar{E})(\nabla^2 E + \frac{1}{F}\vec{\nabla}F \cdot \vec{\nabla}E) - \vec{\nabla}E \cdot \vec{\nabla}E \quad (26)$$

$$N = 6(E + \bar{E})\partial\bar{\partial}K + \frac{3}{E + \bar{E}}(\partial E\bar{\partial}\bar{E} + \bar{\partial}E\partial\bar{E}) = 3pe^{2K} \quad (27)$$

The expressions  $Im\Lambda$  and  $Re\Lambda$  are the imaginary part and the real part of  $\Lambda$  respectively, and  $\hat{\rho}$  and  $\hat{z}$  are the unit vectors in the direction of the  $\rho$  and  $z$  axis respectively. Also in deriving Eq (27) we have used Eqs (22) and (23). Thus we have reduced the solution of the problem to the solution of Eqs (20) - (22), which form a system of four equations for the four unknowns  $F$ ,  $K$ , and  $E = T + i\phi$ . Having solved this system we can use Eqs (23) and (24) to calculate  $p$  and  $\mu$ .

Our system of equations (20) - (24) is simplified considerably in the vacuum case. In this case we have  $p = \mu = 0$  and we can introduced Weyl's

canonical coordinates with  $F = \rho$ . Then Eq (23) is satisfied and Eq (24) implies the relation  $Re\Lambda=0$ . This combined with Eq (20) gives

$$\Lambda = \frac{1}{2}(E + \bar{E})(\nabla^2 E + \frac{1}{\rho}E_\rho) - \vec{\nabla}E \cdot \vec{\nabla}E = 0 \quad (28)$$

which is Ernst's equation. Also Eqs (21) imply the relations

$$K_\rho = \frac{\rho}{4T^2}(T_\rho^2 + \phi_\rho^2 - T_z^2 - \phi_z^2), K_z = \frac{\rho}{2T^2}(T_\rho T_z + \phi_\rho \phi_z) \quad (29)$$

which are consistent (i.e.  $K_{\rho z} = K_{z\rho}$ ) if  $\Lambda = 0$ . Finally Eq (22) is a consequence of Eqs (28) and (29). Therefore in the vacuum case we have to solve Eqs (28) and (29) only.

If we put

$$\ln F = G \quad (30)$$

we get

$$\frac{1}{F}\partial_a\partial_b F = \partial_a\partial_b G + \partial_a G\partial_b G \quad (31)$$

Then Eqs (21) and (22) become

$$\bar{\partial}\bar{\partial}G + \bar{\partial}G\bar{\partial}G - 2\bar{\partial}G\bar{\partial}K + \frac{2}{(E + \bar{E})^2}\bar{\partial}E\bar{\partial}E = 0 \quad (32)$$

$$2\partial\bar{\partial}K - \partial\bar{\partial}G - \partial G\bar{\partial}G + \frac{1}{(E + \bar{E})^2}(\partial E\bar{\partial}\bar{E} + \bar{\partial}E\partial\bar{E}) = 0 \quad (33)$$

To simplify the notation we shall introduce the variables  $\xi$  and  $\eta$  by the relations

$$\xi = \rho + iz \text{ and } \eta = \rho - iz \quad (34)$$

and we shall put

$$2K - G = M \quad (35)$$

Then Eqs (20), (32) and (33) become

$$G_{\eta\eta} - M_\eta G_\eta + \frac{1}{2T^2}(T_\eta^2 + \phi_\eta^2) = 0 \quad (36)$$

$$M_{\eta\xi} - G_\eta G_\xi + \frac{1}{2T^2}(T_\eta T_\xi + \phi_\eta \phi_\xi) = 0 \quad (37)$$

$$\phi_{\eta\xi} + \frac{1}{2}(G_\eta \phi_\xi + G_\xi \phi_\eta) - \frac{1}{T}(T_\eta \phi_\xi + T_\xi \phi_\eta) = 0 \quad (38)$$

Also we can easily write Eqs (23) and (24) in the variables introduced above.

One way of finding solutions is to solve the system of four Eqs (36) - (38) with four unknowns and then use Eqs (23) and (24) to get  $p$  and  $\mu$ . This form of the system of equations i.e. four equations without  $p$  and  $\mu$  and two giving  $p$  and  $\mu$  in term of the four dependent variables is very convenient in the search for Bäcklund transformations [8]. Of course one can approach the problem in a different way.

### 3 Another Form of the System

We shall write now the system we want to solve in a different way. Eqs (32) and (33) become in the variables  $\xi$  and  $\eta$  of Eqs (34)

$$G_{\eta\eta} + G_{\eta}^2 - 2G_{\eta}K_{\eta} + \frac{2}{(E + \overline{E})^2}E_{\eta}\overline{E}_{\eta} = 0 \quad (39)$$

$$2K_{\eta\xi} - G_{\eta\xi} - G_{\eta}G_{\xi} + \frac{1}{(E + \overline{E})^2}(E_{\eta}\overline{E}_{\xi} + E_{\xi}\overline{E}_{\eta}) = 0 \quad (40)$$

Let us put

$$G_{\eta} = N \quad (41)$$

which implies the relation  $G_{\xi} = \overline{N}$  since  $G$  is real. Then solving Eq (39) for  $K_{\eta}$  and using the resulting expression to eliminate  $K_{\eta}$  from Eq (40) we get

$$K_{\eta} = \frac{N_{\eta}}{2N} + \frac{N}{2} + \frac{1}{(E + \overline{E})^2 N} E_{\eta} \overline{E}_{\eta} \quad (42)$$

$$\begin{aligned} \frac{\partial}{\partial \xi} \left\{ \frac{N_{\eta}}{N} + \frac{2}{(E + \overline{E})^2 N} E_{\eta} \overline{E}_{\eta} \right\} &= N \overline{N} \\ - \frac{1}{(E + \overline{E})^2} (E_{\xi} \overline{E}_{\eta} + E_{\eta} \overline{E}_{\xi}) & \end{aligned} \quad (43)$$

Also Eq (20) becomes

$$Im\Lambda = Im\{E_{\xi\eta} + \frac{1}{2}(NE_{\xi} + \overline{N}E_{\eta}) - \frac{2}{E + \overline{E}}E_{\xi}E_{\eta}\} = 0 \quad (44)$$

The  $N$  we get by solving Eqs (43) and (44) must be of the form of Eq (41). But then we get

$$ImN_{\xi} = ImG_{\eta\xi} = 0 \quad (45)$$

since  $G$  is real. We can show that any complex number  $N$  which satisfies the first of Eqs (45) is of the form of Eq (41) with  $G$  real. Also Eq (43) implies the relation

$$\text{Im}K_{\eta\xi} = 0 \quad (46)$$

which means that  $K_\rho$  and  $K_z$  coming from Eqs (42) satisfy the integrability condition  $K_{\rho z} = K_{z\rho}$ . Using Eq (44) we can write Eq (43) in the form

$$\begin{aligned} & \left(\frac{N_\eta}{N}\right)_\xi - N\bar{N} + \frac{2(E_n + \bar{E}_n)}{N(E + \bar{E})^2} \left\{ E_{\xi\eta} + \frac{1}{2}(NE_\xi + \bar{N}E_\eta) \right. \\ & \left. - \frac{2}{E + \bar{E}} E_\xi E_\eta \right\} - \frac{2}{(E + \bar{E})^2} \left( \frac{N_\xi}{N^2} + \frac{\bar{N}}{N} \right) E_\eta \bar{E}_\eta = 0 \end{aligned} \quad (47)$$

Therefore we can get  $N$  and  $E$  by solving Eqs (43) or (47), (44) and the first of (45) and  $K$  from (42).

## References

- [1] D Kramer, H Stephani, M MacCallum and E Herlt *Exact Solutions of Einstein's Field Equations* (Cambridge University Press, London, 1980).
- [2] M Trümper, Z. Naturforschung **22** 1347 (1967)
- [3] F Ernst, Phys. Rev. **167**, 1175 (1968)
- [4] F Ernst, Phys. Rev. **168**, 1415 (1968)
- [5] J Winicour, J. Math. Phys. **16**, 1806 (1975)
- [6] S Bonanos and D Sklavenitis, J. Math. Phys. **26**, 2275 (1985)
- [7] J Ehlers, Dissertation, Hamburg, 1957
- [8] E Kyriakopoulos : *Bäcklund Transformations of Einstein's Field Equations for the Interior of a Uniformly Rotating Stationary, Axisymmetric Perfect Fluid* , NTUA preprint, submitted for publication